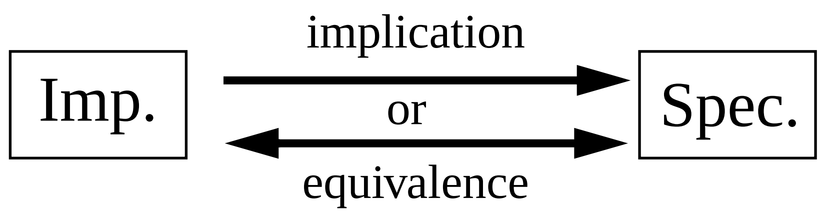
4. Verification by Theorem Proving

Introduction

Theorem Proving

Prove that an implementation satisfies a specification by mathematical reasoning



Implementation and specification expressed as *formulas* in *a formal logic*

Required relationship (logical equivalence/logical implication) described as *a theorem* to be proven within the context of a proof calculus

A proof system:

A set of axioms and inference rules (simplification, rewriting, induction, etc.)

Introduction (cont’d)

Proof checking

purported proof → Proof checker “is this a proof?” → “yes”/“no”

- It is a purely *syntactic* matter to decide whether each theorem is an axiom or follows from previous theorems (axioms) by a rule of inference

Proof generation

purported theorem → Proof generator “prove this theorem” → a proof

- Complete automation generally impossible: theoretical undecidability limitations
- However, a great deal can be automated (decidable subsets, specific classes of applications and specification styles)

First-Order Logic

- *Propositional logic*: reasoning about complete sentences.
- *First-order logic*: also reasoning about *individual objects* and *relationships between them*.

Syntax

- **Objects** (in FOL) are denoted by expressions called ***terms***:

  Constants a, b, c,... ; *Variables* u, v, w,... ;

  $f(t_1, t_2, \ldots, t_n)$ where $t_1, t_2, \ldots, t_n$ are terms and f a *function symbol* of n arguments

- **Predicates**:

  true (T) and *false* (F)

  $p(t_1, t_2, \ldots, t_n)$ where $t_1, t_2, \ldots, t_n$ are *terms* and p a *predicate symbol* of n arguments

- **Formulas**:

  Predicates

  P and Q formulas, then $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$, $P \leftrightarrow Q$ are formulas

  x a variable, P a formula, then $\forall x.P$, $\exists x.Q$ are formulas (x is not free in P, Q)

First-Order Logic (cont'd)

Semantics of a first-order logic formulae G: interpretation for function, constant and predicate symbols in G and assigning values to free variables

First-Order Interpretations (Structures) M: M = (D, I)

- D is a non-empty domain of the structure
- I is an interpretation function, assigns function, constant and predicate symbols:

  (1) For every function symbol f of rank $n>0$, $I(f): D^n \rightarrow D$ is an n-ary function.

  (2) For every constant c, I(c) is an element of D.

  (3) For every predicate symbol P of rank $n\geq 0$, $I(P): D^n \rightarrow \{F, T\}$ is an n-ary predicate.

Evaluation

- For every M, a formula can be evaluated to T or F according to the following rules:

  (1) Evaluate truth values of formulas P and Q, and then the truth values of
  $\neg P$, $P\wedge Q$, $P\vee Q$, $P\rightarrow Q$, $P\leftrightarrow Q$ using propositional logic

  (2) $\forall x.\ P$ evaluates to T if truth value of G is T for every $d\in D$; otherwise, it is F

  (3) $\exists x.P$ evaluates to T if truth value of G is T for at least one $d\in D$; otherwise, it is F

First-Order Logic (cont'd)

Example: $G = \forall x. (P(x) \rightarrow Q(f(x), a))$, with $M=(D, I)$, $D=\{1,2\}$, and I as:

Assignment for a

a
1

Assignment for f

f(1)	f(2)
2	1

Assignment for P and Q

P(1)	P(2)	Q(1,1)	Q(1,2)	Q(2,1)	Q(2,2)
F	T	T	T	F	T

- $x=1$: $P(x)\rightarrow Q(f(x), a) = P(1)\rightarrow Q(f(1), a) = P(1)\rightarrow Q(2, 1) = F\rightarrow F = T$;
- $x=2$: $P(x)\rightarrow Q(f(x), a) = P(2)\rightarrow Q(f(2), a) = P(2)\rightarrow Q(1, 1) = T\rightarrow T = T$.
- Since $P(x)\rightarrow Q(f(x), a)$ is true for all $x \in D$, $\forall x. (P(x)\rightarrow Q(f(x), a))$ is true under M
- M is a model of G ($M \models G$)

(we can also prove that $\exists x. (P(x)\rightarrow Q(f(x), a))$ is true under M)

First-Order Logic (cont'd)

The Validity Problem of FOL

- To decide the validity for formulas of FOL, the truth table method does not work!
- *Reason*: must deal with structures not just truth assignments.
- Structures need not be finite ...

Semi-decidable (partially solvable)

- There is an algorithm which starts with an input, and

  1) if the input is valid then
  it terminates after a finite number of steps, and
  outputs the correct value (Yes or No)

  2) if the input is not valid then it reaches a reject halt or loops forever

Theorem (Church-Turing, 1936)

The validity problem for formulas of FOL is undecidable, but semi-decidable.

- Some subsets of FOL are decidable.

First-Order Logic (cont'd)

Deduction in FOL

Theorem (Gödel, 1931)

- FOL is complete and consistent, i.e., there are complete and consistent deduction systems.

Prenex Normal Forms (PNF): Move quantifiers to the front

$F = \underbrace{(Q_1\ x_1) \ldots (Q_n\ x_n)}_{\text{prefix}}\ G$	every $(Q_i x_i)$ is either $(\forall x_i)$ or $(\exists x_i)$, $i=1,\ldots,n$ G is a formula containing no quatifiers.

- **Theorem:** For every formula P, there exists an equivalent formula Q in prenex form.
- In the proof of this theorem, there is a simple algorithm to convert formulas into PNF.

First-Order Logic (cont'd)

Skolem Standard Forms (SSF)

- **Skolemization**: Remove existential quantifiers

  For each formula F in PNF, define its SSF as the result of applying the following algorithm to F.

  while F contains an existential quantifier **do**
  begin
  Let F have the form $F = \forall x_1 \ldots \forall_n x_n \exists z\, E$ for some E in PNF and $n \geq 0$;
  Let f be a new function symbol of arity n that does not yet occur in F;
  $F := \forall x_1 \ldots \forall_n x_n\, E\, [f(x_1, \ldots, x_n)/z]$; {substitution a/x: a replaces x}
  {$\exists z$ in F is canceled and each occurrence of z in E is substituted by $f(x_1, \ldots, x_n)$}
  end.

Example: $F = \forall x \exists y \exists z\, ((\neg P(x,y) \wedge Q(x,z)) \vee R(x,y,z))$

$$= \forall x \underset{f(x)}{\exists y} \underset{g(x)}{\exists z}\, \underbrace{((\neg P(x,y) \vee R(x,y,z)) \wedge (Q(x,z)) \vee R(x,y,z)))}_{\text{CNF}}$$

The SSF of F is: $\forall x\, ((\neg P(x, f(x)) \vee R(x, f(x), g(x))) \wedge (Q(x, g(x)) \vee R(x, f(x), g(x))))$.

First-Order Logic (cont'd)

Skolem Standard Forms (cont'd)

Theorem: For each formula F in PNF, F is satisfiable iff its SSF is satisfiable.

- Transformation of a formula to Skolem form does not preserve equivalence, because of the new function symbol(s) occurring in the Skolem formula.
- FOL resolution is based on Skolem Standard Forms.

Substitution

- A *substitution* is a finite set of the form $\{t_1/v_1,..., t_n/v_n\}$, where every v_i is a variable, every t_i is a term different from v_i, and no two elements in the set have the same variable after the stroke symbol.
- Examples: $\psi = \{f(z)/x, y/z\}$ and $\theta = \{a/x, g(y)/y, f(g(b))/z\}$.
- Let $\theta = \{t_1/v_1,..., t_n/v_n\}$ be a substitution and E be an expression.
  $E\theta$ is an expression obtained by replacing *simultaneously* each occurrence of variable v_i, $0 \le i \le n$, in E by term t_i.
- $E\theta$ is an *instance* of E.
- Example: $\theta=\{a/x,f(b)/y,c/z\}$ and $E=P(x,y,z)$; $E\theta=P(a,f(b),c)$.

First-Order Logic (cont'd)

- Substitutions $\theta = \{t_1/x_1, \ldots, t_n/x_n\}$ and $\lambda = \{u_1/y_1, \ldots, u_m/y_m\}$

  C*omposition of* θ *and* λ is the substitution $\theta \bullet \lambda$ obtained from

  $\{t_1\lambda/x_1, \ldots, t_n\lambda/x_n, u_1/y_1, \ldots, u_m/y_m\}$ by deleting

  (1) any element $t_j\lambda/x_j$, for which $t_j\lambda = x_j$, and

  (2) any element u_i/y_i such that $y_i \in \{x_1, \ldots, x_n\}$

- Example: $\theta = \{f(y)/x, z/y\}$ and $\lambda = \{a/x, b/y, y/z\}$
  $\{f(y)/x, \cancel{y/y}, \cancel{a/x}, \cancel{b/y}, y/z\} \Rightarrow \theta \bullet \lambda = \{f(b)/x, y/z\}$

Unification

- While carrying out proofs, we have to unify (match) two or more expressions
- Must find a substitution that can make several expressions identical
- Substitution θ is a *unifier* for $\{E_1, \ldots, E_k\}$ iff $E_1\theta = E_2\theta = \ldots = E_k\theta$
- $\{E_1, \ldots, E_k\}$ is *unifiable* if there is a unifier for it
- Example: $\{P(a, y), P(x, f(b))\}$ is unifiable with $\theta = \{a/x, f(b)/y\}$
- Unifier σ for $S = \{E_1, \ldots, E_k\}$ of expressions is *most general unifier* iff for each unifier θ of S there is a substitution λ such that $\theta = \sigma \bullet \lambda$

First-Order Logic (cont'd)

Unification (cont'd)   Unification plays a key role in proof systems.

- Basic idea: expressions P(a) and P(x) are not identical and

  {a, x} is the disagreement pair - try to eliminate it by unification

  Since x is a (universally quantified) variable, it can be replaced by a and the disagreement thus eliminated!

- *Unification Algorithm (Robinson)*

  k: = 0; $\theta_k = \varnothing$, $W_k := W$ {A non-empty set of literals}

  while $|W_k| > 1$ **do**

  begin

  Scan terms in W_k from left to right, until the first position is found where in at least two literals (say, L_1 and L_2) the corresponding symbols are different;

  if none of these symbols is a variable **then** output "non-unifiable" and halt;

  else Let x be the variable and t the other term;

  if x occurs in t **then** output "non-unifiable" and halt

  else $\theta_{k+1} := \theta_k \bullet \{t/x\}$, $W_{k+1} := W_k\{t/x\}$;

  k:=k+1

  end

First-Order Logic (cont'd)

Unification (cont'd)

- Example: $W=\{\neg P(f(z, g(a,y)), h(z)), \neg P(f(f(u,v),w), h(f(a,b)))\}$

Step 1: $\{ \neg P(f(z, g(a,y)), h(z)), \quad \neg P(f(f(u,v),w), h(f(a,b))) \}$ $\quad \theta_1 := \{f(u,v)/z\}$

Step 2: $\{\neg P(f(f(u,v), g(a,y)), h(f(u,v))), \neg P(f(f(u,v),w), h(f(a,b)))\}$ $\quad \theta_2 := \theta_1 \bullet \{g(a,y)/w\}$

Step 3: $\{\neg P(f(f(u,v), g(a,y)), h(f(u,v))), \neg P(f(f(u,v), g(a,y)), h(f(a,b)))\}$ $\quad \theta_3 := \theta_2 \bullet \{a/u\}$

Step 4: $\{\neg P(f(f(u,v), g(a,y)), h(f(a,v))), \neg P(f(f(u,v), g(a,y)), h(f(a,b)))\}$ $\quad \theta_4 := \theta_3 \bullet \{b/v\}$

$W = \{\neg P(f(f(a,b),g(a,y)),h(f(a,b)))\}$ $\quad \theta_4 := \{f(u,v)/z, g(a,y)/w, a/u, b/v\}$

Unification Theorem (Robinson): Each unifiable set of literals has the most general unifier.

Higher-Order Logic

- *First-order logic*: only domain variables can be quantified.
- *Second-order logic*: quantification over subsets of variables (i.e., over predicates).
- *Higher-order logics*: quantification over arbitrary predicates and functions.

Higher-Order Logic

- Variables can be functions and predicates,
- Functions and predicates can take functions as arguments and return functions as *values*,
- Quantification over functions and predicates.

  Since arguments and results of predicates and functions can themselves be predicates or functions, this imparts a **first-class status** to functions, and allows them to be manipulated just like *ordinary values*

Example 1: (mathematical induction)

$\forall P. [P(0) \land (\forall n. P(n) \longrightarrow P(n+1))] \longrightarrow \forall n.P(n)$ (Impossible to express it in FOL)

Example 2: Function Rise defined as Rise (c, t) = $\neg c(t) \land c(t+1)$

Rise expresses the notion that a signal *c* rises at time *t*.

Signal is modeled by a function c: $N \rightarrow \{F,T\}$, passed as argument to Rise.

Result of applying Rise to c is a function: $N \rightarrow \{F,T\}$.

Higher-Order Logic (cont'd)

Advantage: high expressive power!

Disadvantages:

- Incompleteness of a sound proof system for most higher-order logics
- **Theorem** (Gödel, 1931)
  There is no complete deduction system for the second-order logic.
- Reasoning more difficult than in FOL, need ingenious inference rules and heuristics.
- Inconsistencies can arise in higher-order systems if semantics not carefully defined

  "*Russell Paradox*":

  Let P be defined by $P(Q) = \neg Q(Q)$. By substituting P for Q, leads to $P(P) = \neg P(P)$,
  ($P: bool \rightarrow bool$, $Q: bool \rightarrow bool$) contradiction!
- Introduction of "types" (syntactical mechanism) is effective against certain inconsistencies.
- Use *controlled form of logic and inferences* to minimize the risk of inconsistencies, while gaining the benefits of powerful representation mechanism.
- Higher-order logic increasingly popular for hardware verification!

Theorem Proving Systems

- Automated deduction systems (e.g. Prolog)
  - full automatic, but only for a decidable subset of FOL
  - speed emphasized over versatility
  - often implemented by ad hoc decision procedures
  - often developed in the context of AI research

- Interactive theorem proving systems
  - semi-automatic, but not restricted to a decidable subset
  - versatility emphasized over speed
  - in principle, a complete proof can be generated for every theorem

Some theorem proving systems:

Boyer-Moore (first-order logic)

HOL (higher-order logic)

PVS (higher-order logic)

Lambda (higher-order logic)

Boyer-Moore (Nqthm)

- Developed at University of Texas and later CLI
- Quantifier-free first-order logic.
- Powerful built-in heuristics; user must find a sequence of lemmas that permits to prove the desired theorem with available heuristics
- Collection of LISP programs that permit the user to axiomatize inductively constructed data types, define recursive functions, and (inductively) prove theorems about them
- Process of proof generation is not fully automatic; user assistance for setting up intermediate lemmas and definitions
- A number of verification application including microprocessors
- For more information: `http://www.cli.com/`

ACL2

- Developed at CLI
- ACL2 is a mathematical logic together with a mechanical theorem prover to help reason in the logic
- The logic is just a subset of applicative Common Lisp
- The theorem prover is an “industrial strength” version of the Boyer-Moore theorem prover, Nqthm
- Models of all kinds of computing systems can be built in ACL2, just as in Nqthm, even though the formal logic is Lisp
- Once built, an ACL2 model of a system can be *executed* in Common Lisp
- ACL2 can also be used to prove theorems about the model
- For more information: `http://www.cs.utexas.edu/users/moore/acl2/`

PVS

- PVS (Prototype Verification System) developed at SRI
- The specification language of PVS is based on classical, typed higher-order logic
- The primitive inferences include propositional and quantifier rules, induction, rewriting, and *decision procedures* for linear arithmetic
- The implementations of these primitive inferences are optimized for large proofs: E.g., propositional simplification uses BDDs, and auto-rewrites are cached for efficiency
- User-defined procedures can combine these primitive inferences to yield higher-level proof strategies
- PVS includes a *BDD-based decision procedure* for relational Mu-calculus: experimental integration of theorem proving and CTL model checking
- Proofs are developed interactively by combining high-level inference procedures:
- For more information: `http://www.csl.sri.com/pvs.html`

Lambda

- Commercial tool by Abstract Hardware Ltd. (UK)
- Verification and synthesis tool based on high-order logic theorem proving
- Specification in predicate logic and expressed in the L2 language (based on SML, Standard Meta Language)
- Specification can be executed using the “Animator” tool
- Interactive *correct-by-construction* synthesis using

  - transformations by applying rewriting rules

  - partitioning

  - instantiating and interconnecting components

  - scheduling operations, and allocating resources (even for pipelined designs)

- Backtracking to a preceding design and exploration of alternatives
- Reasoning over a mix of timing scales, e.g., clock ticks, frame periods, pipeline insertion
- Output current state of the design (subset of L2) in **VHDL** and produce control microcode
- Complex properties can be stated and proven as formulas to be satisfied by the design
- For more information: `http://www.ahl.co.uk`

HOL

- HOL (Higher-Order Logic) developed at University of Cambridge
- Interactive environment (in ML, Meta Language) for machine assisted theorem proving in higher-order logic (a proof assistant)
- Steps of a proof are implemented by applying inference rules chosen by the user; HOL checks that the steps are safe
- All inferences rules are built on top of eight primitive inference rules
- Mechanism to carry out backward proofs by applying built-in ML functions called *tactics* and *tacticals*
- By building complex tactics, the user can customize proof strategies
- Numerous applications in software and hardware verification
- Large user community
- For more information: `http://www.cl.cam.ac.uk/Research/HVG/HOL/` or `http://lal.cs.byu.edu/lal/hol-documentation.html`

<u>Note</u>: we will now focus on HOL!

HOL Theorem Prover

- Logic is strongly typed (type inference, abstract data types, polymorphic types, etc.)
- It is sufficient for expressing most ordinary mathematical theories (the power of this logic is similar to set theory)
- HOL provides considerable built-in theorem-proving infrastructure:

  - a powerful *rewriting* subsystems

  - *library* facility containing useful theories and tools for general use

  - *Decision procedures* for tautologies and semi-decision procedure for linear arithmetic provided as libraries
- The primary interface to HOL is the functional programming language ML
- Theorem proving tools are functions in ML (users of HOL build their own application-specific theorem proving infrastructure by writing programs in ML)
- Many versions of HOL:

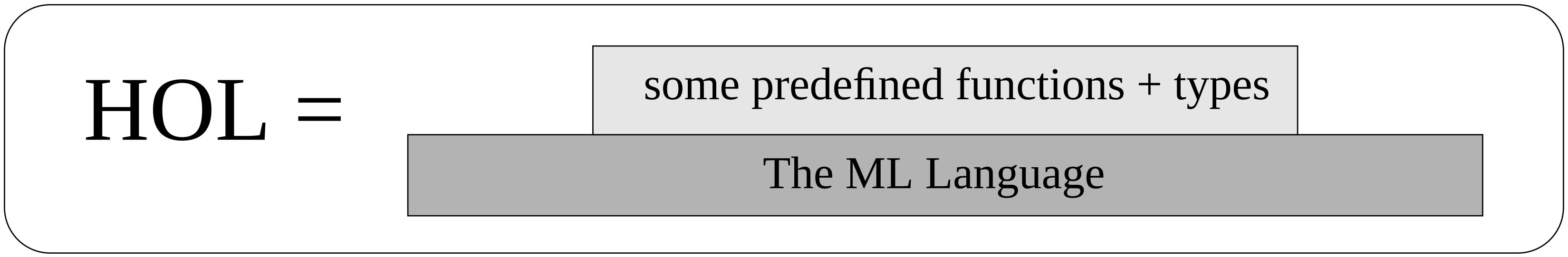
  - HOL88: Classic ML (from LCF);

  - HOL90: Standard ML

  - HOL98: Moscow ML

HOL Theorem Prover (cont'd)

- HOL and ML



- The HOL systems can be used in two main ways:
  - for directly proving theorems: when higher-order logic is a suitable specification language (e.g., for hardware verification and classical mathematics)
  - as embedded theorem proving support for application-specific verification systems when specification in specific formalisms needed to be supported using customized tools.
- The approach to mechanizing formal proof used in HOL is due to Robin Milner.

  He designed a system, called LCF: Logic for Computable Functions. (The HOL system is a direct descendant of LCF.)

HOL Theorem Prover (cont'd)

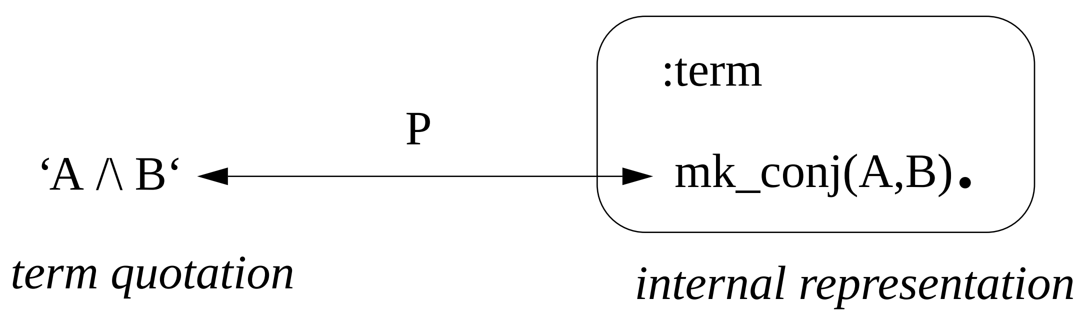
- How the logic is embedded in ML:

logic	terms	types	theorems
ML data type	**`:term`**	**`:hol_type`**	**`:thm`**

- Terms are represented by values of the ML abstract data type **`:term`**

```
- P ‘T /\ F ==> T‘;
val it = ‘T/\ F ==> T‘ : term
```

- The quotation parser and pretyyprinter:



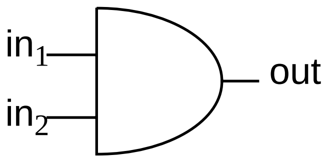
Specification in HOL

- **Functional description**:

  express output signal as function of input signals, e.g.:

  AND gate:

  out = **and** $(in_1, in_2) = (in_1 \wedge in_2)$



- **Relational (predicate) description**:

  gives relationship between inputs and outputs in the form of a predicate (a Boolean function returning “true” of “false”), e.g.:

  AND gate:

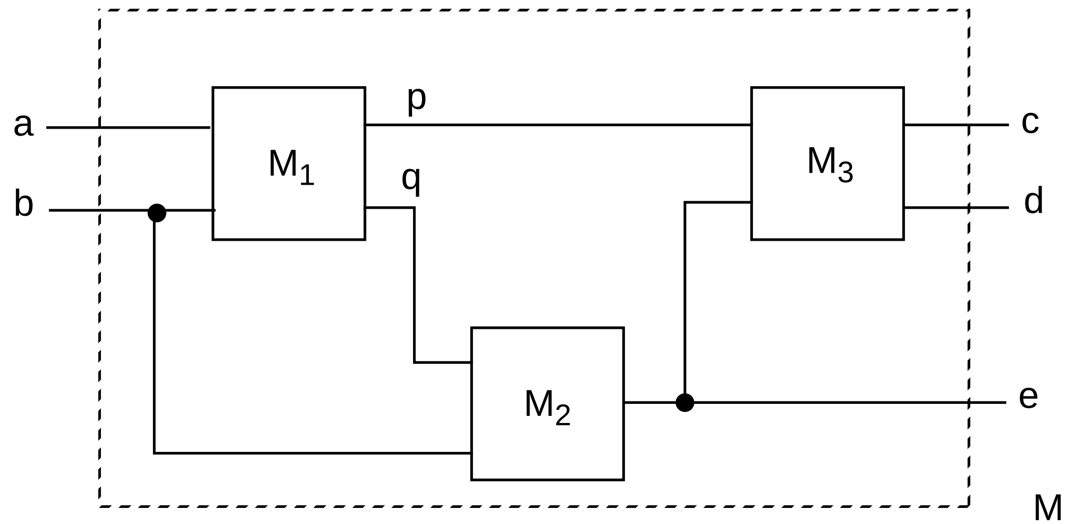
  AND $((in_1, in_2),(out))$:= out $=(in_1 \wedge in_2)$

Notes:

- functional descriptions allow recursive functions to be described. They cannot describe bi-directional signal behavior or functions with multiple feed-back signals, though
- relational descriptions make no difference between inputs and outputs
- Specification in HOL will be a combination of predicates, functions and abstract types

Specification in HOL

Network of modules



- conjunction "$\wedge$" of implementation module predicates

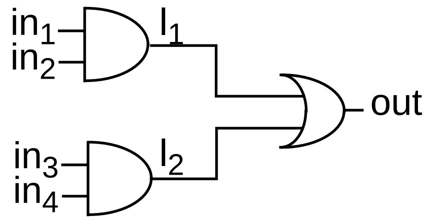
$$M(a, b, c, d, e) := M_1(a, b, p, q) \wedge M_2(q, b, e) \wedge M_3(e, p, c, d)$$

- hide internal lines (p,q) using **existential quantification**

$$M(a, b, c, d, e) := \exists p\, q.\ M_1(a, b, p, q) \wedge M_2(q, b, e) \wedge M_3(e, p, c, d)$$

Specification in HOL

Combinational circuits



SPEC $(in_1, in_2, in_3, in_4, out) :=$

$$out = (in_1 \wedge in_2) \vee (in_3 \wedge in_4)$$

IMPL $(in_1, in_2, in_3, in_4, out) :=$

$$\exists\, l_1\, l_2.\ \mathbf{AND}\,(in_1, in_2, l_1) \wedge \mathbf{AND}\,(in_3, in_4, l_2) \wedge \mathbf{OR}\,(l_1, l_2, out)$$

where **AND** $(a, b, c) := (c = a \wedge b)$
OR $(a, b, c) := (c = a \vee b)$

Note: a functional description would be:

IMPL $(in_1, in_2, in_3, in_4, out) :=$

$$out = (\text{or } (\text{and } (in_1, in_2), \text{and } (in_3, in_4))$$

where **and** $(in_1, in_2) = (in_1 \wedge in_2)$
or $(in_1, in_2) = (in_1 \vee in_2)$

Specification in HOL

Sequential circuits

- Explicit expression of time (discrete time modeled as natural numbers).
- Signals defined as functions over time, e.g. type: (**nat** $\rightarrow$ **bool**) or (**nat** $\rightarrow$ **bitvec**)
- Example: D-flip-flop (latch):

  DFF (in, out):= (out (0) = F) $\wedge$ ($\forall$ t. out (t+1) = in (t))

  in and *out* are functions of time *t* to boolean values: type (nat $\rightarrow$ bool)

- Notion of time can be added to combinational circuits, e.g., AND gate

  AND (in_1, in_2, out):= $\forall$ t. out (t) = (in_1(t) $\wedge$ in_2(t))

- Temporal operators can be defines as predicates, e.g.:

  EVENTUAL sig t_1 = $\exists$ t_2. ($t_2 > t_1$) $\wedge$ sig t_2

  meaning that signal “sig” will eventually be true at time $t_2 > t_1$.

  Note: This kind of specification using existential quantified time variables is useful to describe asynchronous behavior

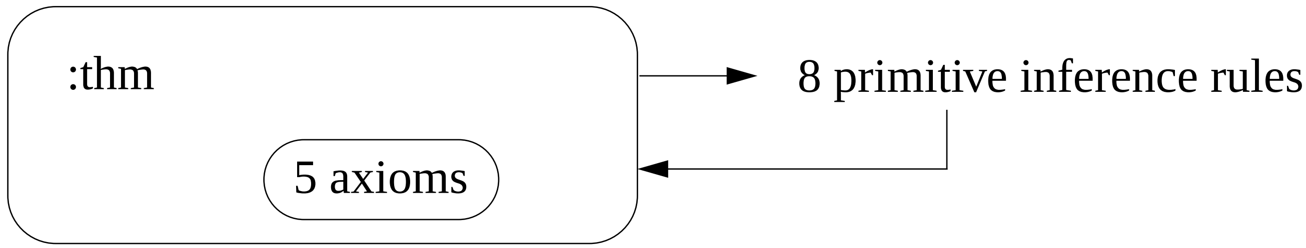
HOL Proof Mechanism

- A formal proof is a sequence, each of whose elements is
  - either an *axiom*
  - or follows from earlier members of the sequence by a *rule of inference*
- A *theorem* is the last element of a proof
- A *sequent* is written: $\Gamma \vdash P$, where Γ is a *set of assumptions* and P is the *conclusion*
- In HOL, this consists in applying ML functions representing rules of inference to axioms or previously generated theorems
- The sequence of such applications directly correspond to a proof
- A value of *type* **`thm`** can be obtained either
  - directly (as an axiom)
  - by computation (using the built-in functions that represent the inference rules)
- ML typechecking ensures these are the only ways to generate a thm:

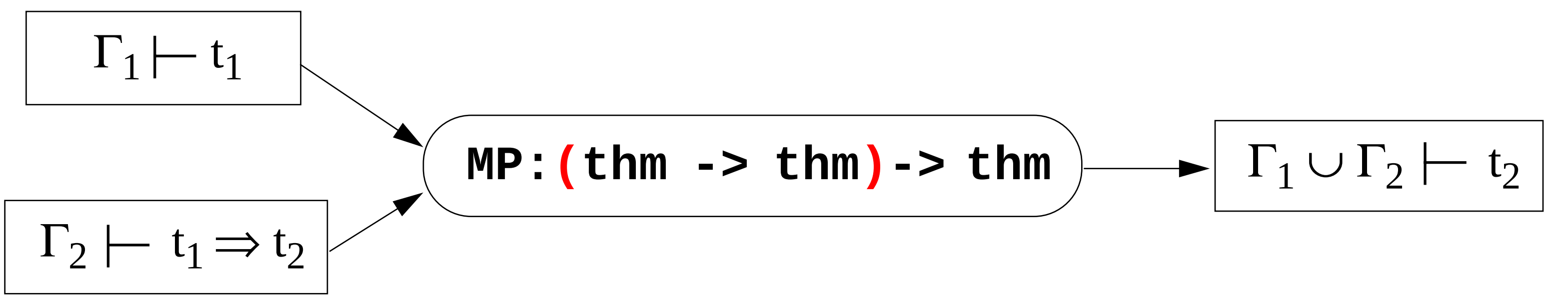
  All theorems must be proved!

HOL Proof System

- In the core of HOL:



- An *inference rule*: as an ML function that returns a theorem as a result
- Example: *modus ponens* in HOL,



- The function returns only objects of type **`thm`** that logically follow by the inference rule

Primitive Rules

- All theorems in HOL are ultimately proved using only the primitive inference rule:

ASSUME:

$$\frac{}{\{t\} \vdash t}$$

REFL:

$$\frac{}{\vdash t = t}$$

BETA CONVERSION:

$$\frac{}{\vdash (\lambda x.\ t_1)\ t_2 = t_1[t_2/x]}$$

ABSTRACTION:

$$\frac{\Gamma \vdash t_1 = t_2}{\Gamma \vdash (\lambda x.\ t_1) = (\lambda x.\ t_2)} \quad (\text{x not free in } \Gamma)$$

INST_TYPE:

$$\frac{\Gamma \vdash t}{\Gamma \vdash t[\sigma_1, ..., \sigma_n/\alpha_1, ..., \alpha_n]}$$

DISCHARGE:

$$\frac{\Gamma \vdash t_2}{\Gamma - \{t_1\} \vdash t_1 \Rightarrow t_2}$$

MODUS PONENS:

$$\frac{\Gamma_1 \vdash t_1 \Rightarrow t_2\ ,\ \Gamma_2 \vdash t_1}{\Gamma_1 \cup \Gamma_2 \vdash t_2}$$

SUBST:

$$\frac{\Gamma_1 \vdash t_1 = t_1',\ .....,\ \Gamma_n \vdash t_n = t_n',\ \Gamma \vdash t}{\Gamma_1 \cup ..... \cup \Gamma_n \cup \Gamma \vdash t[t_1, ..., t_n/t_1', ..., t_n']}$$

Basic Rewriting Rules

- Rewriting is done:
  - with all the supplied equations
  - on all subterms of the theorem to be rewritten
  - repeatedly, until no rewrite rule applies

- Rewriting rules:

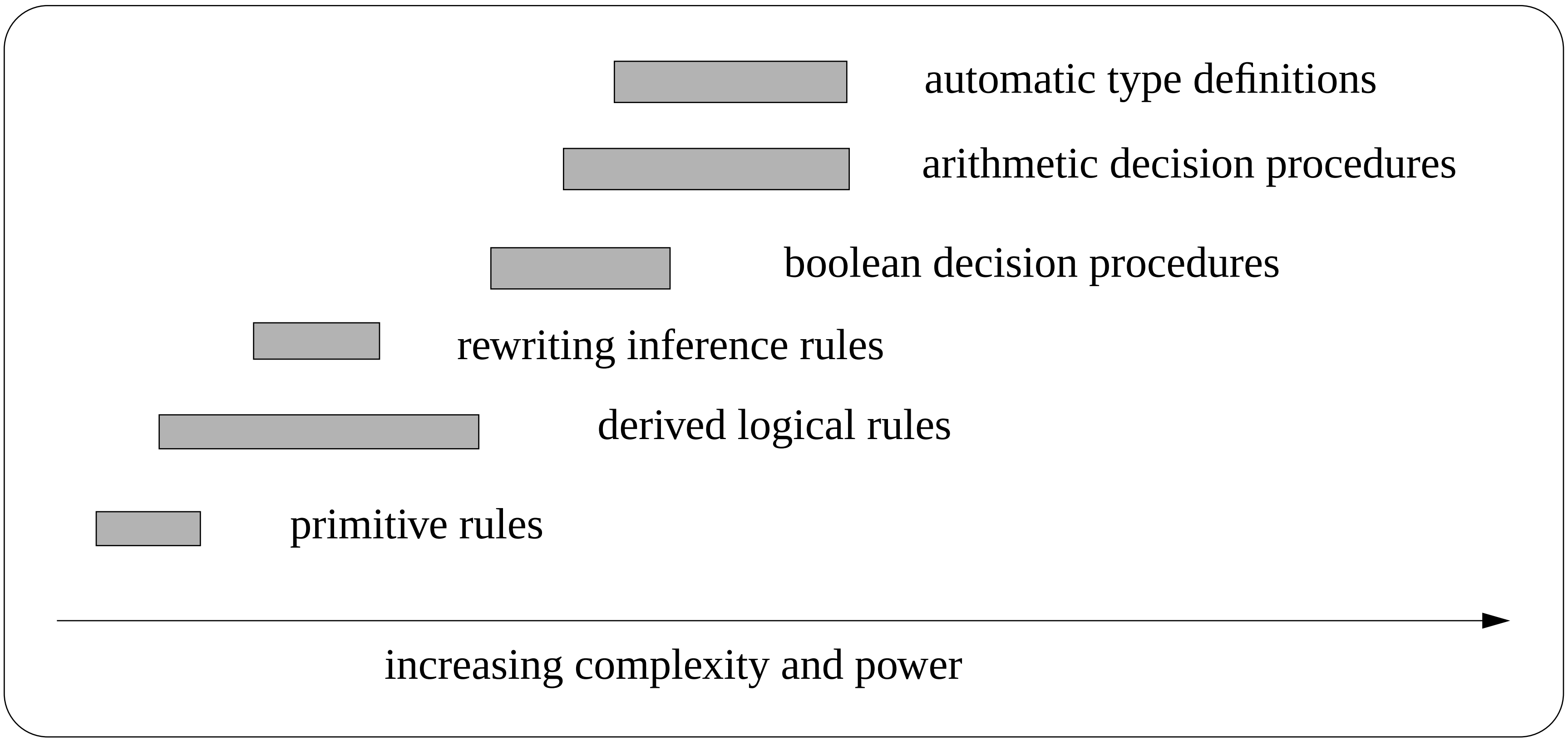
Rewriting rules: **`thm list -> thm -> thm`**

theorems to be rewritten

list of equational theorems to be used as left-to-right rewrite rules

Built-in Derived Rules

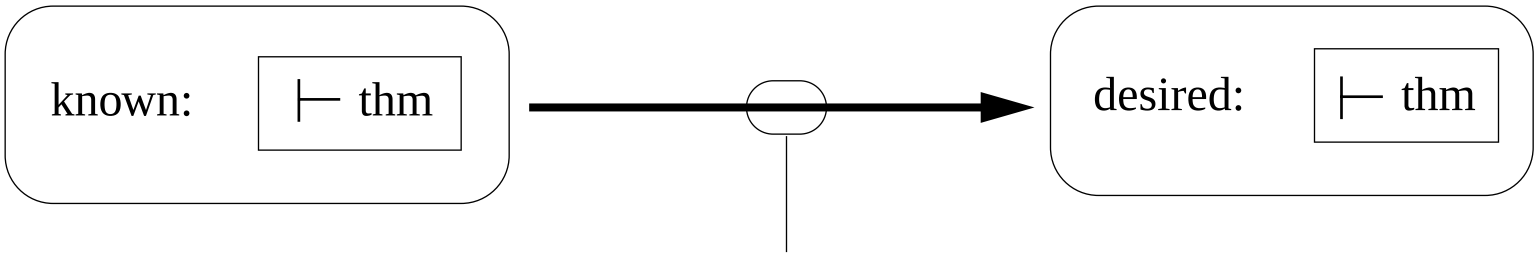
- There is a wide range of derived inference rules built into the system:



- To become an expert HOL user, one should continuously learn new rules and proof techniques

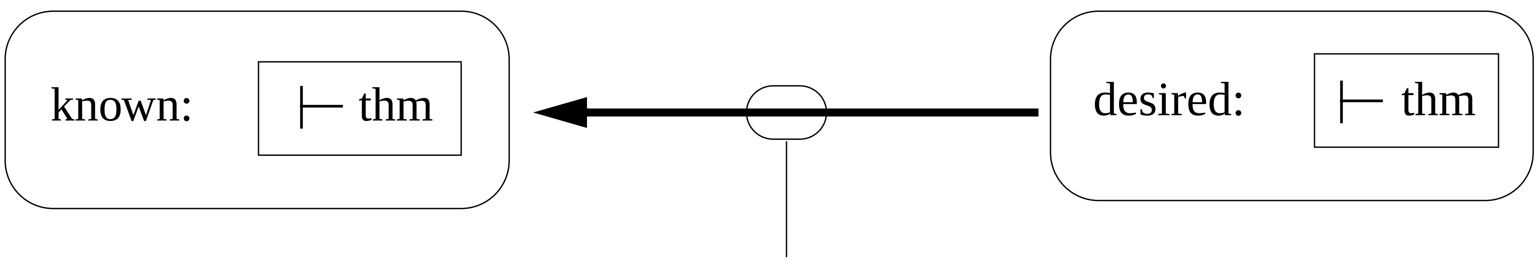
Proof Styles in HOL

- **Forward** proof style:



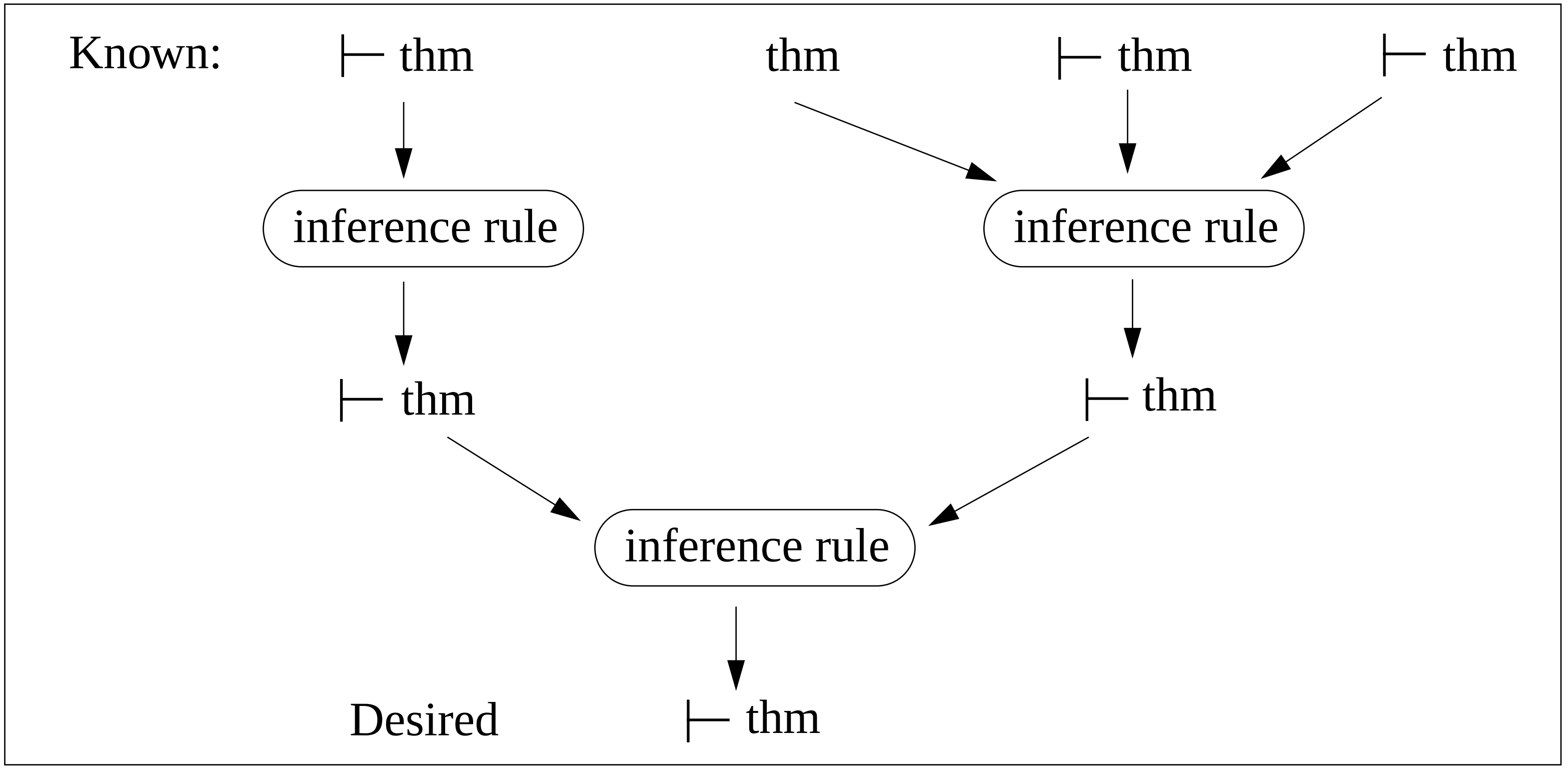
apply inference rules to infer the desired
theorem from already proved theorems

- Goal-directed (or **Backward**) proof style:



reduce goal to be proved to simpler and simpler
subgoals until known theorems reached

Forward Proof in HOL



- can be millions of (primitive) inferences long
- usually not natural for “one-off” proofs
- but essential for tool building

Backward Proof in HOL

- Goals are represented by values of ML type

goal: $\underbrace{\text{term list}}_{\text{assumptions}} * \underbrace{\text{term}}_{\text{conclusion}}$

$P_1, ..., P_n \vdash P \xrightarrow{\text{solves}} ([P_1, ..., P_n], P)$

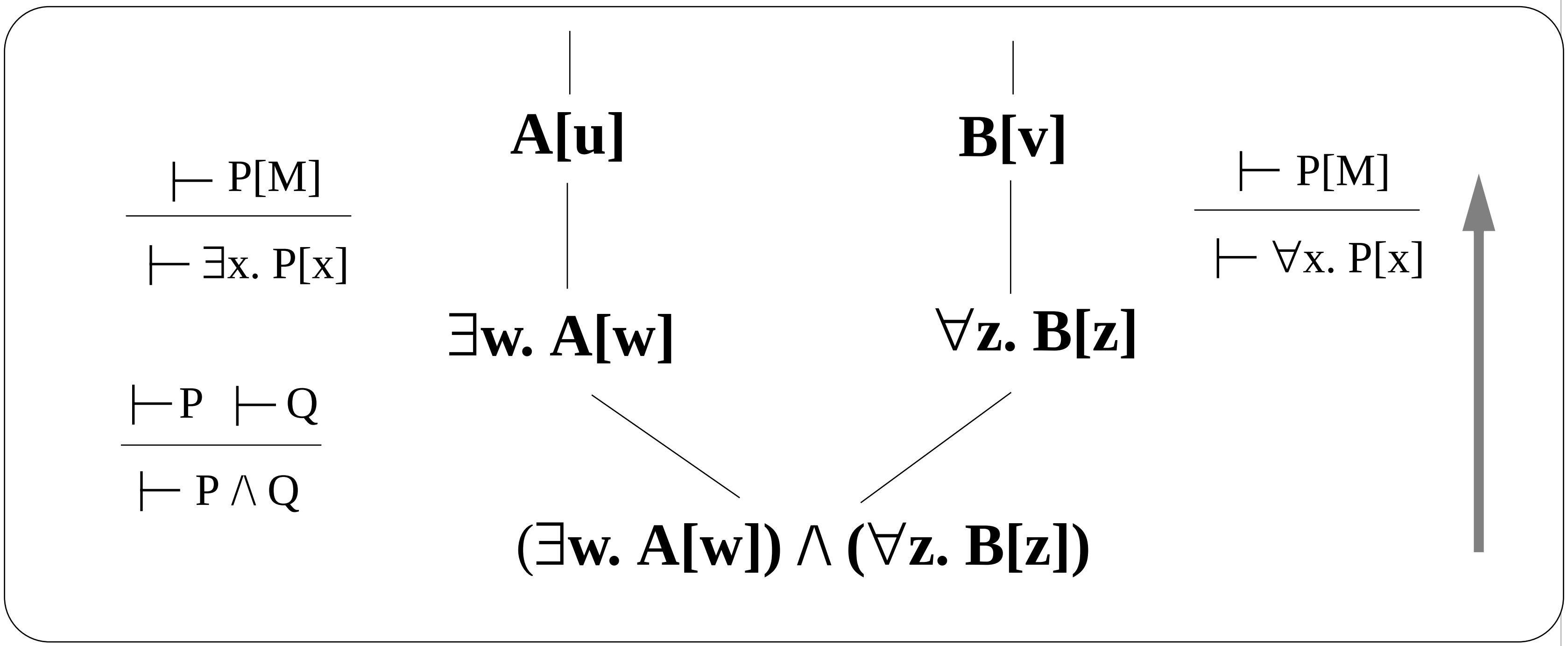
: thm : goal

- Goal-directed proof in HOL:

$$\begin{tikzcd}
\text{|- thm} \arrow[d, equal] & & \vdash \text{goal} \arrow[ll, "\textit{user}\text{ works backwards}"'] \\
\text{|- thm} \arrow[rr, "\textit{machine}\text{ builds the proof}"] & & \vdash \text{thm} \arrow[u, "\text{solves}"']
\end{tikzcd}$$

Backward Proof

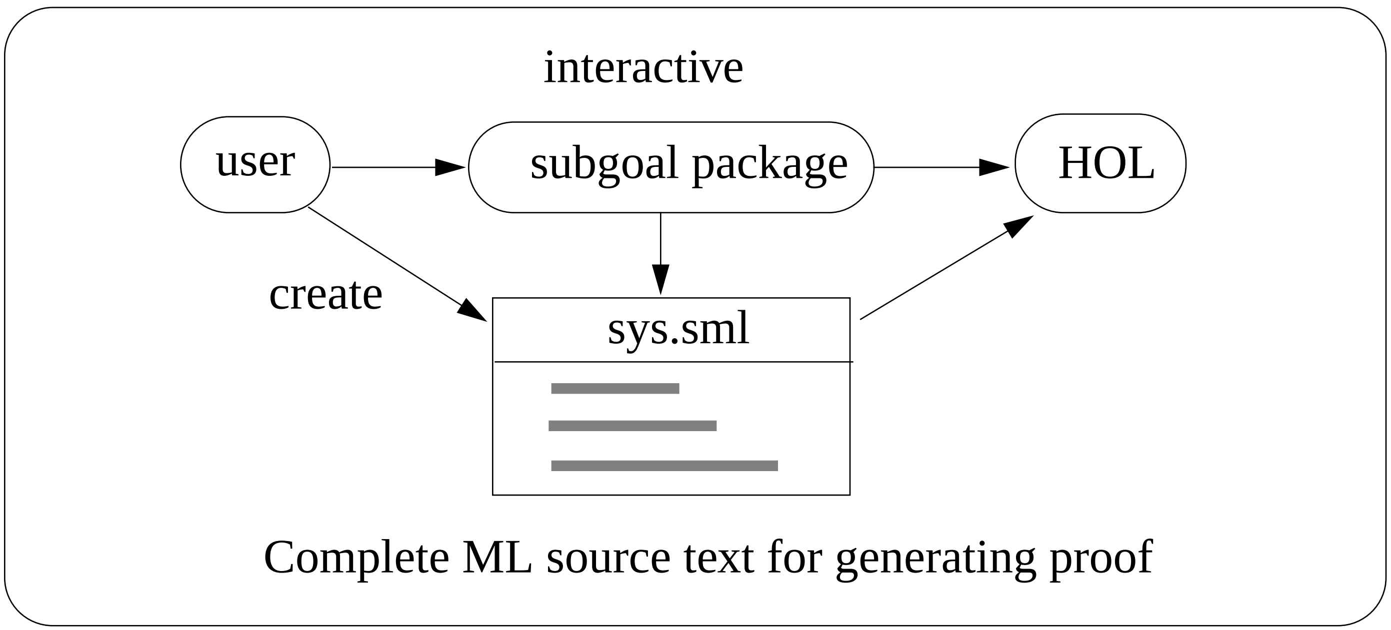
- Example:



- Reduction of a goal to *subgoals* is justified by an inference in the “opposite direction”.

The Subgoal Package

- HOL has a subgoal package for finding tactic proofs interactively
- The subgoal package:
    - maintains a *stack* of subgoals to be proved
    - provides functions that operate on these subgoals
- The subgoal package is for finding the schema of the proof:



HOL Tactics

- Tactic is a function:

$$T : \texttt{goal -> } \underbrace{\texttt{goal list}}_{\text{subgoals}}\ \underbrace{\texttt{(thm list -> thm)}}_{\text{justification}}$$

- Suppose that for a given goal g: $T(g) = ([g_1, ..., g_n], f)$
- If the theorems $\Gamma_1 \vdash P_1, ..., \Gamma_n \vdash P_n$ solve the goals $g_1, ..., g_n$, then $f([\Gamma_1 \vdash P_1, ..., \Gamma_n \vdash P_n])$ should solve the original goal g.
- In a picture:

$$\begin{tikzcd}
([g_1, ..., g_n], f) \arrow[d, equal] & g \arrow[l, "\text{Tactics}"] \\
{[\Gamma_1 \vdash P_1, ..., \Gamma_n \vdash P_n]} \arrow[r, "f"] & \Gamma \vdash P \arrow[u, "\text{solves}"']
\end{tikzcd}$$

HOL Tactics (Examples)

$$\frac{A \vdash g}{A \cup \{t\} \vdash g} \quad \text{ASSUM\_TAC } (A \vdash t)$$

$$\frac{A \lor B}{A} \quad \text{DISJ1\_TAC} \qquad \frac{A \lor B}{B} \quad \text{DISJ2\_TAC}$$

$$\frac{A \vdash t_1 = t_2}{A \vdash t_1 \Longrightarrow t_2 \qquad A \vdash t_2 \Longrightarrow t_1} \quad \text{EQ\_TAC}$$

$$\frac{A \vdash !x.\ P}{A \vdash P[x'/x]} \quad \text{GEN\_TAC}$$

Verification Methodology in HOL

1. Establish a formal specification (predicate) of the intended behavior (SPEC)

2. Establish a formal description (predicate) of the implementation (IMP), including:

   - behavioral specification of all sub-modules

   - structural description of the network of sub-modules

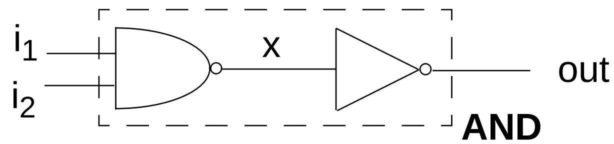
3. Formulation of a proof goal, either

   - IMP $\Rightarrow$ SPEC (proof of implication), or

   - IMP $\Leftrightarrow$ SPEC (proof of equivalence)

4. Formal verification of above goal using a set of inference rules

Example 1: Logic AND

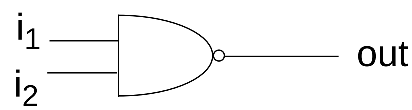


AND Specification:

AND_SPEC $(i_1, i_2, out) := out = i_1 \wedge i_2$

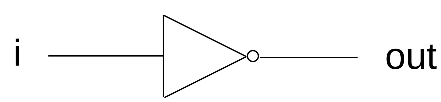
NAND specification:

NAND $(i_1, i_2, out) := out = \neg(i_1 \wedge i_2)$



NOT specification:

NOT $(i, out) := out = \neg i$



AND Implementation:

AND_IMPL $(i_1, i_2, out) := \exists x.$ NAND $(i_1, i_2, x) \wedge$ NOT (x, out)

Logic AND (cont'd)

Proof Goal:

$\forall$ i_1, i_2, out. AND_IMPL(i_1,i_2,out) $\Rightarrow$ ANDSPEC(i_1,i_2,out)

Proof (forward)

AND_IMP(i_1,i_2,out) {from above circuit diagram}

$\vdash$ $\exists$x.NAND (i_1,i_2,x) $\wedge$ NOT (x,out) {by def. of AND impl}

$\vdash$ NAND (i_1,i_2,***x***) $\wedge$ NOT(***x***,out) {strip off "$\exists$x."}

$\vdash$ NAND (i_1,i_2,***x***) {left conjunct of line 3}

$\vdash$ x =$\neg$($i_1$$\wedge$$i_2$) {by def. of NAND}

$\vdash$ NOT (*x*,out) {right conjunct of line 3}

$\vdash$ out =$\neg$x {by def. of NOT}

$\vdash$ out =$\neg$($\neg$($i_1$$\wedge$$i_2$) {substitution, line 5 into 7}

$\vdash$ out =($i_1$$\wedge$$i_2$) {simplify,$\neg\neg$ t=t}

$\vdash$ AND (i_1,i_2,out) {by def. of AND spec}

$\vdash$ AND_IMPL (i_1,i_2,out) $\Rightarrow$ AND_SPEC (i_1,i_2,out)

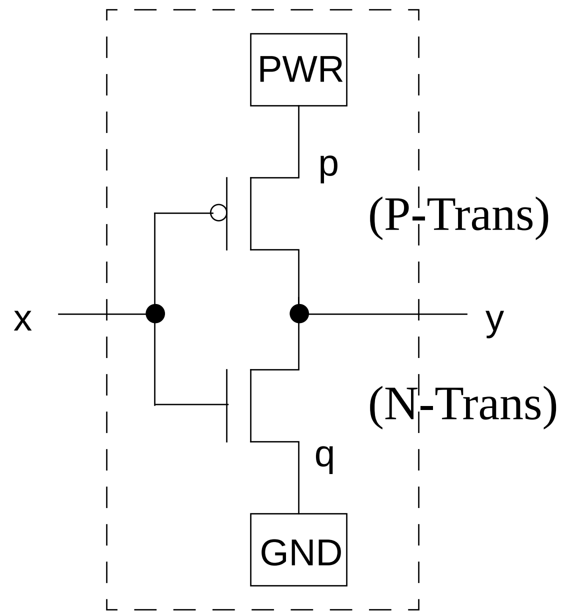
Q.E.D.

Example 2: CMOS-Inverter

Specification (black-box behavior)

Spec(x,y):= $(y = \neg x)$

Implementation



Basic Modules Specs

PWR(x):= $(x = T)$

GND(x):= $(x = F)$

N-Trans(g,x,y):= $(g \Rightarrow (x = y))$

P-Trans(g,x,y):= $(\neg g \Rightarrow (x = y))$

Implementation (network structure)

$$
\begin{aligned}
\mathrm{Impl}(x,y) := \exists\, p\, q.\ & \mathrm{PWR}(p) \wedge \\
& \mathrm{GND}(q) \wedge \\
& \text{N-Tran}(x,y,q) \wedge \\
& \text{P-Tran}(x,p,y)
\end{aligned}
$$

Proof goal

$$\forall\, x\, y.\ \mathrm{Impl}(x,y) \Leftrightarrow \mathrm{Spec}(x,y)$$

Proof (forward)

$$
\begin{aligned}
\mathrm{Impl}(x,y) := \exists\, p\, q.\ & (p = T) \wedge \\
& (q = F) \wedge \qquad \text{(substitution of the definition of PWR and GND)} \\
& \text{N-Tran}(x,y,q) \wedge \\
& \text{P-Tran}(x,p,y)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Impl}(x,y) := \exists\, p\, q.\ & (p = T) \wedge \\
& (q = F) \wedge \qquad \text{(substitution of p and q in P-Tran and N-Tran)} \\
& \text{N-Tran}(x,y,F) \wedge \\
& \text{P-Tran}(x,T,y)
\end{aligned}
$$

$$\begin{aligned}\mathrm{Impl}(x,y) := {} & (\exists\, p.\ p = T) \wedge \\ & (\exists\, q.\ q = F) \wedge \\ & \text{N-Tran}(x,y,F) \wedge \\ & \text{P-Tran}(x,T,y)\end{aligned}$$

(use Thm: "$\exists$ a. t1 $\wedge$ t2 = ($\exists$ a. t_1) $\wedge$ t_2" if a is free in t_2)

$$\begin{aligned}\mathrm{Impl}(x,y) := {} & T \wedge \\ & T \wedge \\ & \text{N-Tran}(x,y,F) \wedge \\ & \text{P-Tran}(x,T,y)\end{aligned}$$

(use Thm: "($\exists$ a. a=T) = T" and "($\exists$ a. a=F) = T")

$$\begin{aligned}\mathrm{Impl}(x,y) := {} & \text{N-Tran}(x,y,F) \wedge \\ & \text{P-Tran}(x,T,q)\end{aligned}$$

(use Thm: "x $\wedge$ T = x")

$$\begin{aligned}\mathrm{Impl}(x,y) := {} & (x \Rightarrow (y = F)) \wedge \\ & (\neg x \ \Rightarrow (T = y))\end{aligned}$$

(use def. of N-Tran and P-Tran)

$$\begin{aligned}\mathrm{Impl}(x,y) := {} & (\neg x \vee (y = F)) \wedge \\ & (x \ \vee\ (T = y))\end{aligned}$$

(use "(a $\Rightarrow$ b) = ($\neg$ a $\vee$ b)")

Boolean simplifications:

$\mathrm{Impl}(x,y) := (\neg x \wedge x) \vee (\neg x \wedge (T = y)) \vee ((y = F) \wedge x) \vee ((y = F) \wedge (T = y))$

$\mathrm{Impl}(x,y) := F \vee (\neg x \wedge (T = y)) \vee ((y = F) \wedge x) \vee F$

$\mathrm{Impl}(x,y) := (\neg x \wedge (T = y)) \vee ((y = F) \wedge x)$

Case analysis x=T/F

$x=T: \mathrm{Impl}(T,y) := (F \wedge (T = y)) \vee ((y = F) \wedge T)$

$x=F: \mathrm{Impl}(F,y) := (T \wedge (T = y)) \vee ((y = F) \wedge F)$

$x=T: \mathrm{Impl}(T,y) := (y = F)$

$x=F: \mathrm{Impl}(F,y) := (T = y)$

$\overset{!}{=}$

Case analysis on **Spec**:

$x=T: \mathrm{Spec}(T,y) := (y=F)$

$x=F: \mathrm{Spec}(F,y) := (y=T)$

Conclusion: $\vdash \mathrm{Spec}(x,y) \Leftrightarrow \mathrm{Impl}(x,y)$

Abstraction Forms

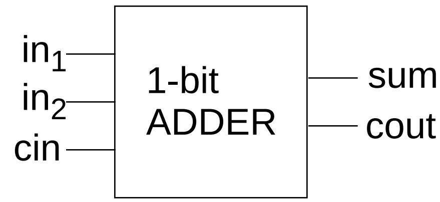
- **Structural abstraction:** only the behavior of the external inputs and outputs of a module is of interest (abstracts away any internal details)

- **Behavioral abstraction:** only a specific part of the total behavior (or behavior under specific environment) is of interest

- **Data abstraction:** behavior described using abstract data types (e.g. natural numbers instead of Boolean vectors)

- **Temporal abstraction:** behavior described using different time granularities (e.g. refinement of instruction cycles to clock cycles)

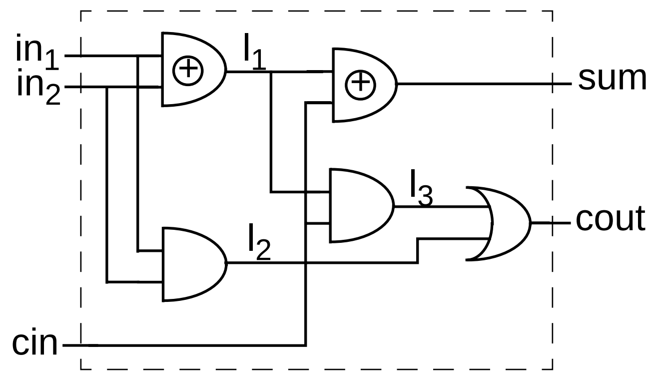
Example 3: 1-bit Adder



Specification:

ADDER_SPEC (in_1:**nat**, in_2:**nat**, cin:**nat**, sum:**nat**, cout:**nat**):= $in_1 + in_2 + cin = 2 * cout + sum$

Implementation:



<u>Note</u>: Spec is a **structural abstraction** of Impl.

1-bit Adder (cont'd)

Implementation:

$$
\begin{aligned}
&\text{ADDER\_IMPL}(in_1{:}\mathbf{bool},\ in_2{:}\mathbf{bool},\ cin{:}\mathbf{bool},\ sum{:}\mathbf{bool},\ cout{:}\mathbf{bool}) := \\
&\quad \exists\, l_1\, l_2\, l_3.\ \ \mathrm{EXOR}\ (in_1, in_2, l_1) \wedge \\
&\qquad\qquad \mathrm{AND}\ (in_1, in_2, l_2) \wedge \\
&\qquad\qquad \mathrm{EXOR}\ (l_1, cin, sum) \wedge \\
&\qquad\qquad \mathrm{AND}\ (l_1, cin, l_3) \wedge \\
&\qquad\qquad \mathrm{OR}\ (l_2, l_3, cout)
\end{aligned}
$$

Define a **data abstraction function** (**bn: bool** $\rightarrow$ **nat**) needed to relate Spec variable types (nat) to Impl variable types (bool):

$$
bn(x) := \begin{cases} 1, & \text{if } x = T \\ 0, & \text{if } x = F \end{cases}
$$

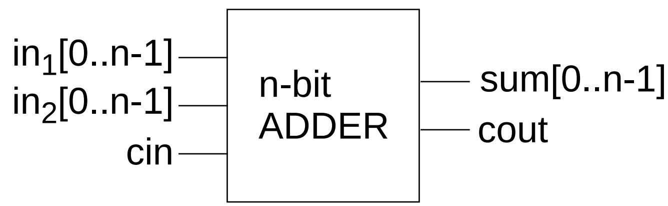
Proof goal:

$$
\begin{aligned}
&\forall\, in_1, in_2, cin, sum, cout. \\
&\quad \text{ADDER\_IMPL}\ (in_1, in_2, cin, sum, cout) \\
&\qquad \Rightarrow \text{ADDER\_SPEC}\ (\mathbf{bn}(in_1), \mathbf{bn}(in_2), \mathbf{bn}(cin), \mathbf{bn}(sum), \mathbf{bn}(cout))
\end{aligned}
$$

Verification of Generic Circuits

- used in datapath design and verification
- **idea**: verify **n**-bit circuit then specialize proof for specific value of **n**, (i.e., once proven for **n**, a simple instantiation of the theorem for any concrete value, e.g. 32, gets a proven theorem for that instance).
- use of induction proof

Example: N-bit Adder

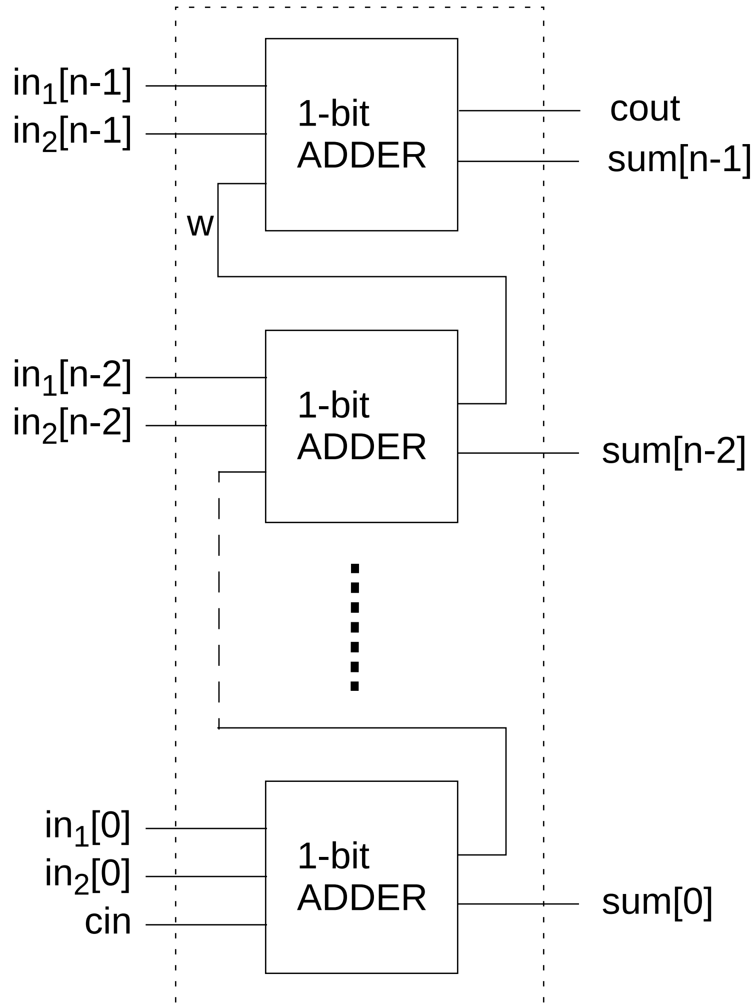


Specification

N-ADDER_SPEC (**n**,in_1,in_2,cin,sum,cout):= (in_1 + in_2 + cin = $2^{n+1} *$ cout + sum)

Example 4: N-bit Adder

Implementation



N-bit Adder (cont'd)

Implementation

- recursive definition:

  N-ADDER_IMP(n,in_1[0..n-1],in_2[0..n-1],cin,sum[0..n-1],cout):=

  $\exists$ w. N-ADDER_IMP(n-1,in_1[0..n-2],in_2[0..n-2],cin,sum[0..n-2],w) $\wedge$

  N-ADDER_IMP(1,in_1[n-1],in_2[n-1],w,sum[n-1],cout)

- Note: N-ADDER_IMP(1,in_1[i],in_2[i],cin,sum[i],cout) =

  ADDER_IMP(in_1[i],in_2[i],cin,sum[i],cout)

- Data abstraction function (**vn: bitvec** $\rightarrow$ **nat**) to relate bit vctors to natural numbers:

  vn(x[0]):= bn(x[0])

  vn(x[0,n]):= $2^n * $ bn(x[n]) + vn(x[0,n-1]

Proof goal:

$\forall$ **n**, in_1, in_2, cin, sum, cout.

N-ADDER_IMP(n,in_1[0..n-1],in_2[0..n-1],cin,sum[0..n-1],cout)

$\Rightarrow$ N-ADDER_SPEC(n,**vn**(in_1[0..n-1]),**vn**(in_2[0..n-1]),**vn**(cin),**vn**(sum[0..n-1]),**vn**(cout))

can be **instantiated** with **n = 32:**

$\forall$ in_1, in_2, cin, sum, cout.

N-ADDER_IMP(in_1[0..31],in_2[0..31],cin,sum[0..31],cout)

$\Rightarrow$ N-ADDER_SPEC(**vn**(in_1[0..31]),**vn**(in_2[0..31]),**vn**(cin),**vn**(sum[0..31]),**vn**(cout))

N-bit Adder (cont'd)

Proof by induction over n:

- basis step:

  N-ADDER_IMP(0,in_1[0],in_2[0],cin,sum[0],cout)

  $\Rightarrow$ N-ADDER_SPEC(0,**vn**(in_1[0]),**vn**(in_2[0]),**vn**(cin),**vn**(sum[0]),**vn**(cout))

- induction step:

  [N-ADDER_IMP(n,in_1[0..n-1],in_2[0..n-1],cin,sum[0..n-1],cout) $\Rightarrow$

  N-ADDER_SPEC(n,**vn**(in_1[0..n-1]),**vn**(in_2[0..n-1]),**vn**(cin),**vn**(sum[0..n-1]),**vn**(cout))**]**

  $\Rightarrow$

  [N-ADDER_IMP(n+1,in_1[0..n],in_2[0..n],cin,sum[0..n],cout) $\Rightarrow$

  N-ADDER_SPEC(n+1,**vn**(in_1[0..n]),**vn**(in_2[0..n]),**vn**(cin),**vn**(sum[0..n]),**vn**(cout))**]**

Notes:

- basis step is equivalent to 1-bit adder proof, i.e.

  ADDER_IMP(in_1[0],in_2[0],cin,sum[0],cout)

  $\Rightarrow$ ADDER_SPEC(**bn**(in_1[0]),**bn**(in_2[0]),**bn**(cin),**bn**(sum[0]),**bn**(cout))

- induction step needs more creativity and work load!

Practical Issues of Theorem Proving

No fully automatic theorem provers. All require human guidance in indirect form, such as:

- When to delete redundant hypotheses, when to keep a copy of a hypothesis
- Why and how (order) to use lemmas, what lemma to use is an art
- How and when to apply rules and rewrites
- Induction hints (also nested induction)
- Selection of proof strategy, orientation of equations, etc.
- Manipulation of quantifiers (forall, exists)
- Instantiation of specification to a certain time and instantiating time to an expression
- Proving lemmas about (modulus) arithmetic
- Trying to prove a false lemma may be long before abandoning

Conclusions

Advantages of Theorem Proving

- High abstraction and expressive notation
- Powerful logic and reasoning, e.g., induction
- Can exploit hierarchy and regularity, puts user in control
- Can be customized with tactics (programs that build larger proofs steps from basic ones)
- Useful for specifying and verifying parameterized (generic) datapath-dominated designs
- Unrestricted applications (at least theoretically)

Limitations of Theorem Proving:

- Interactive (under user guidance): use many lemmas, large numbers of commands
- Large human investment to prove small theorems
- Usable only by experts: difficult to prove large / hard theorems
- Requires deep understanding of the both the design and HOL (while-box verification)
- must develop proficiency in proving by working on simple but similar problems.
- Automated for narrow classes of designs

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